WAVELET–GALERKIN METHOD FOR NUMERICAL SOLUTION OF PARTIAL DIFFERENTIAL EQUATION

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF

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BY

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DECLARATION

I, the undersigned, declare that the work contained in this thesis entitled **"Wavelet-Galerkin Method for Numerical Solution of Partial Differential Equation"** in partial fulfilment of the requirement for the award of the degree of Master of Science, submitted in the Department of Mathematics, National Institute of Technology, Rourkela, is entirely my own work and has not previously in its entirely or quoted have been indicated and approximately acknowledged by complete references.

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CERTIFICATE

This is to certify that the thesis entitled **"Wavelet-Galerkin Method for Numerical Solution of Partial Differential Equation"** submitted by Mr. Ashish Kumar Kalia, Roll No.412MA2066, for the award of the degree of Master of Science from National Institute of Technology, Rourkela, is absolutely best upon her own work under the guidance of Prof. S.Saha Ray. The results embodied in this thesis are new and neither this thesis nor any part of it has been submitted for any degree/diploma or any academic award anywhere before.

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ABSTRACT

In recent years wavelets are given much attention in many branches of science and technology due to its comprehensive mathematical power and application potential. The advantage of wavelet techniques over finite difference or element method is well known. The objective of this work is to implementing the Wavelet-Galerkin method for approximating solutions of differential equations. In this paper, we elaborate the wavelet techniques and apply the Galerkin method procedure to analyse one dimensional wave equation as a test problem using fictitious boundary approach.

The sections of this thesis will include defining wavelets and their scaling functions. This will give the reader valued insight about wavelets. Following this will be a section defining the Daubechies wavelet and its scaling function. This section comprises some subsection about computing the scaling function and its derivative and integral. The purpose of this section will be to give the reader an understanding how scaling function and its derivative are computed. Next will be a section on multiresolution analysis and its properties. The next section give information about the 2-term connection coefficients.

The main focus of this work will be to solve the one dimensional wave equation using fictitious boundary approach and made a comparison between the exact and approximate solution which gives the accuracy and efficiency of this method.

CONTENTS

CHAPTER-1

1. Introduction:

The wavelet based numerical solution has recently developed the theory and applications of partial differential equations. There are many techniques used to approximate the solutions of partial differential equations. Among these approximations wavelet-Galerkin method is the most frequently used technique in this days.

Wavelet analysis is a numerical concept which allows to represent a function in terms of a set of bases functions, which called wavelets. Wavelets are localized both in location and scale. Wavelets used in this method are mostly compact supported wavelets [1].

Daubechies constructed a family of orthonormal bases of compactly supported wavelets for the space of square-integrable functions, $L^2(R)$ [2]. Due to their useful properties, such as compact support, orthogonality, exact representation of polynomials to a certain degree, and ability to represent functions at different levels of resolution, Daubechies wavelets have gained interest in the numerical solutions of ordinary and partial differential equations by several researchers [3, 4, 5].

Daubechies wavelets as basis for the Galerkin method used to solve differential equations which require a computational domain of simple shape. This has been possible due to the remarkable work by Latto et al. [6], Xu et al. [7], Williams et al. and Amartunga et al. [8]. Yet there is more difficulties in dealing with the boundary conditions. So far problems having periodic boundary conditions or periodic distribution have been dealt successfully. In this paper, fictitious boundary approach with Dirichlet boundaries has been applied to solve the 1D wave equation.

Wavelet: Wavelets be a family of oscillatory functions with zero mean constructed from translation and dialation of a single function ψ, called the mother wavelet. We denoted wavelets by $\psi_{a,b}(t), a \in R \setminus \{0\}$ and $b \in R$.

$$\psi_{a,b}(t) = (a)^{-1/2}\psi\left(\frac{t-b}{a}\right) \qquad a, b \in R, a \neq o. \tag{1.1}$$

Here a is called a scaling parameter which measures the degree of compression or scale and b is a translation parameter which determines the time location of the wavelet.

Wavelet satisfies the condition.

$$\int_{-\infty}^{\infty} \frac{|\,\widehat{\psi}(w)\,|^2}{|w|} dw < \infty \tag{1.2}$$

Where $\widehat{\psi}(w)$ is the Fourier transformation of $\psi(t)$.

CHAPTER-2

2. Daubechies Wavelet:

Daubechies introduced the class of compactly supported wavelet bases in 1988. This means that they have non zero values within a finite interval and have a zero value everywhere else. This is the reason that it is useful for representing the solution of differential equation. Daubechies wavelets are an orthonormal basis for functions in the square integrable functions $L^2(R)$[1,2]. Daubechies defined scaling function as

$$\varphi(x)=\sum_{k=0}^{N-1} a_k \varphi(2x-k) \tag{2.1}$$

Where $a_k(k=0,1,\dots N-1)$ called the filter coefficients, N (an even integer) denotes the genus of the Daubechies wavelet. The functions generated with these coefficients will have $\operatorname{supp}(\varphi)=[0,N-1]$ and $\lfloor N/2-1 \rfloor$ vanishing wavelet moments. Daubechies defined wavelet function as

$$\psi(x)=\sum_{k=N}^{1}(-1)^k a_{1-k}\varphi(2x-k) \tag{2.2}$$

The filter coefficients satisfy the following conditions:

$$\begin{aligned}\sum_{k=0}^{N-1} a_k &= 2 \\ \sum_{k=0}^{N-1} a_k a_{k-m} &= 2\delta_{0,m}\end{aligned} \tag{2.3}$$

Where $\delta_{0,m}$ is the Kronecker delta.

We have computed the Daubechies filter coefficients for N= 4, 6, 8, 10, 12, 14, 16 in the Table given below:

Table 1.1: Daubechies Wavelet Filter coefficient a_k

K	N=4	N=6	N=8	N=10	N=12	N=14	N=16
0	0.6830	0.4705	0.3258	0.2264	0.1577	0.1101	0.0769
1	1.1830	1.1411	1.0109	0.8539	0.6995	0.5608	0.4425
2	0.3169	0.6504	0.8922	1.0243	1.0623	1.0312	0.9555
3	-0.1830	-0.1909	0.0396	0.1958	0.4458	0.6644	0.8278
4		-0.1208	-0.2645	-0.3426	-0.3199	-0.2035	-0.0224
5		0.0498	0.0436	-0.0456	-0.1835	-0.3168	-0.4016
6			0.0465	0.1097	0.1378	0.1008	0.0007
7			-0.0149	-0.0088	0.0389	0.1140	0.1821
8				-0.0178	-0.0446	-0.0538	-0.0246
9				0.0047	0.0008	-0.0234	-0.0624
10					0.0068	0.0177	0.0198
11					-0.0015	0.0006	0.0124
12						-0.0025	-0.0069
13						0.0005	-0.0006
14							0.0009
15							-0.0002

2.1. Computation of scaling function [9]:

Computation of scaling function is more over same as eigen value and eigen vector problem.

Example: DAUBECHIES-6 (*N*= 6)

For *N=6* Eq. (2.1) gives

$$\begin{aligned}
\varphi(0.0) &= a_0\varphi(0)+a_1\varphi(-1)+a_2\varphi(-2)+a_3\varphi(-3)+a_4\varphi(-4)+a_5\varphi(-5)\\
\varphi(0.5) &= a_0\varphi(1)+a_1\varphi(0)+a_2\varphi(-1)+a_3\varphi(-2)+a_4\varphi(-3)+a_5\varphi(-4)\\
\varphi(1.0) &= a_0\varphi(2)+a_1\varphi(1)+a_2\varphi(0)+a_3\varphi(-1)+a_4\varphi(-2)+a_5\varphi(-3)\\
\varphi(1.5) &= a_0\varphi(3)+a_1\varphi(2)+a_2\varphi(1)+a_3\varphi(0)+a_4\varphi(-1)+a_5\varphi(-2)\\
\varphi(2.0) &= a_0\varphi(4)+a_1\varphi(3)+a_2\varphi(2)+a_3\varphi(1)+a_4\varphi(0)+a_5\varphi(-1)\\
\varphi(2.5) &= a_0\varphi(5)+a_1\varphi(4)+a_2\varphi(3)+a_3\varphi(2)+a_4\varphi(1)+a_5\varphi(0) \qquad (2.1.1)\\
\varphi(3.0) &= a_0\varphi(6)+a_1\varphi(5)+a_2\varphi(4)+a_3\varphi(3)+a_4\varphi(2)+a_5\varphi(1)\\
\varphi(3.5) &= a_0\varphi(7)+a_1\varphi(6)+a_2\varphi(5)+a_3\varphi(4)+a_4\varphi(3)+a_5\varphi(2)\\
\varphi(4.0) &= a_0\varphi(8)+a_1\varphi(7)+a_2\varphi(6)+a_3\varphi(5)+a_4\varphi(4)+a_5\varphi(3)\\
\varphi(4.5) &= a_0\varphi(9)+a_1\varphi(8)+a_2\varphi(7)+a_3\varphi(6)+a_4\varphi(5)+a_5\varphi(4)\\
\varphi(5.0) &= a_0\varphi(10)+a_1\varphi(9)+a_2\varphi(8)+a_3\varphi(7)+a_4\varphi(6)+a_5\varphi(5)
\end{aligned}$$

Since supp(φ) = $[0, N-1]=[0,5]$

Therefore $\varphi(-1)=\varphi(-2)=\varphi(-3)=\varphi(-4)=\varphi(-5)=\varphi(6)=\varphi(7)=\varphi(8)=\varphi(9)=\varphi(10)=0$

and $a_k \neq 0,\ \ k=0,1,\ldots,5 \Rightarrow \varphi(0) = \varphi(5) = 0$

Now Eq. (2.1.1) becomes

$$\varphi(0) \quad = 0$$

$$\varphi(0.5) = a_0\varphi(1)$$

$$\varphi(1.0) = a_0\varphi(2) + a_1\varphi(1)$$

$$\varphi(1.5) = a_0\varphi(3) + a_1\varphi(2) + a_2\varphi(1)$$

$$\varphi(2.0) = a_0\varphi(4) + a_1\varphi(3) + a_2\varphi(2) + a_3\varphi(1)$$

$$\varphi(2.5) = a_1\varphi(4) + a_2\varphi(3) + a_3\varphi(2) + a_4\varphi(1)$$

$$\varphi(3.0) = a_2\varphi(4) + a_3\varphi(3) + a_4\varphi(2) + a_5\varphi(1)$$

$$\varphi(3.5) = a_3\varphi(4) + a_4\varphi(3) + a_5\varphi(2)$$

$$\varphi(4.0) \ = a_4\varphi(4) + a_5\varphi(3)$$

$$\varphi(4.5) = a_5\varphi(4)$$

$$\varphi(5.0) \quad = 0$$

Write non-trivial equations in matrix form.

$$\begin{bmatrix} \varphi(0.5) \\ \varphi(1.0) \\ \vdots \\ \varphi(4.5) \end{bmatrix} = \begin{bmatrix} 0 & a_0 & 0 & 0 & \dots 0 & 0 \\ 0 & a_1 & 0 & a_0 & \dots 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots a_5 & 0 \end{bmatrix} \begin{bmatrix} \varphi(0.5) \\ \varphi(1.0) \\ \vdots \\ \varphi(4.5) \end{bmatrix}$$

$$\vec{\varphi} = A\vec{\varphi}$$

Eigenvalue - Eigenvector Problem

$$\lambda\vec{v} = A\vec{v}$$ Where $\lambda = 1$ and $\vec{v}$ unknown

Due to partition of unity $\sum v_k = 1$, so

$$\varphi(0.5) + \varphi(1) + \ \dots \ + \varphi(4.5) = 1 \qquad (2.1.2)$$

Eigenvalues of matrix A are 1, 0.499997, 0.470467, 0.270284, 0.250002, 0.0498175, 0, 0, 0, 0 and 0.

Using Eq. (2.1.2) we get, the scaling functions, for the eigenvector components, at the dyadic points $x = k/2^j$ which is given in the table below:

Table 1.2: Daubechies scaling function $\varphi(x)$

x	$\varphi(x)$
0	0
0.5	0.605174940
1	1.286334358
1.5	0.441123730
2	-0.385832129
2.5	-0.014971937
3	0.095266576
3.5	-0.031540759
4	0.004234285
4.5	0.000210940
5	0

2.2. Computation of derivatives of $\varphi(x)$ [10]:

Define the nth derivative of the scaling function $\varphi(x)$ as $\varphi^{(n)}(x)$:

$$\frac{d^n\varphi(x)}{dx^n} = \varphi^{(n)}(x) \qquad n = 0,1,\ldots,N/2-1 \tag{2.2.1}$$

In general, scaling function and its derivatives do not have closed-form solution. They can be obtained from the refinement equations recursively. Applying the two-scale relation Eq. (2.1) to Eq. (2.2.1), we have

$$2^n \sum_{k=0}^{N-1} a_k \varphi^{(n)}(2x-k) = \varphi^{(n)}(x) \tag{2.2.2}$$

We can compute the values of $\varphi^{(n)}(x)$ at all dyadic points $x = k/2^j$, $j = 1, 2, \ldots$ on that condition the values of $\varphi^{(n)}(k)$, $k = 1, 2, \cdots N-2$, are given. So we substitute $x = 1, 2, \cdots N-2$ into the two-scale relation of $\varphi^{(n)}(x)$ Eq. (2.2.2) to get the following homogeneous linear system of equations

$$A\varphi = 2^{-n}\,\varphi \tag{2.2.3}$$

where $\varphi = [\varphi^{(n)}(1), \varphi^{(n)}(2), \ldots, \varphi^{(n)}(N-2)]^T$

$$A = [a_{2i-k}]_{1\le i,k\le N-2} \tag{2.2.4}$$

Eq. (2.2.3) indicates that the unknown vector φ is the eigenvector of the matrix A corresponding to the eigenvalue 2^{-n}. As in all eigenvalue problems, the solution to the system

$$(A - 2^{-n}\, I)\varphi = 0 \tag{2.2.5}$$

is not unique. Thus a normalizing condition is required in order to determine a unique eigenvector. Some calculation shows that

$$\sum_{k=1}^{N-2} (-k)^n \varphi^{(n)}(k) = n! \tag{2.2.6}$$

After computing the values of $\varphi^{(n)}(x)$ for $x = 1, 2, \cdots N-2$, we can use the refinement equation once again

$$\varphi^{(n)}\left(\frac{k}{2^j}\right) = 2^n \sum_{m=0}^{N-1} a_m \varphi^{(n)}\left(\frac{k}{2^{j-1}} - m\right) \tag{2.2.7}$$

to determine the values of $\varphi^{(n)}(x)$ at $x = k/2^j$ for $k = 1, 3, 5, \ldots, 2^j(N-1)-1$an $j = 1, 2, \ldots$ when $n = 0$, the two-scale relation Eq. (2.2.2) is exactly the same as Eq. (2.1) and the above algorithm becomes the determination of the scaling function $\varphi(x)$ itself.

2.3. Define the integrals of $\varphi(x)$ as $r(x)$ [10]:

$$r(x) = \int_o^x \varphi(y)dy \tag{2.2.8}$$

Applying the two scale relation Eq. (2.1) to Eq. (2.2.8), we have

$$\begin{aligned} r(x) &= \int_o^x \sum_{K=0}^{N-1} a_k \varphi(2y - k)dy \\ &= \frac{1}{2}\sum_{K=0}^{N-1} a_k \int_o^x \varphi(2y - k)d(2y - k) \qquad (2.2.9) \\ &= \frac{1}{2}\sum_{K=0}^{N-1} a_k r(2x - k) \end{aligned}$$

It is known that $\varphi(x)$ vanishes for $x \geq N-1$ and that $\int_0^{N-1} \varphi(x)dx = 1$(from property scaling function), there is obtained $r(N-1) = 1$. The values of $r(x)$ for $x = 1, 2, \cdots N-2$ can be determined from the following linear system of equations, which can arrived at $x = 1, 2, \cdots N-2$:

$$(A - 2I)\mathrm{R} = B \tag{2.2.10}$$

In which $B = [b_1\, b_2 \cdots b_{N-1}]^T$

$$b_i = \sum_{\substack{2i-k \geq N-1 \\ k=0,1,\cdots N-1}} a_k r(2i - k) \tag{2.2.11}$$

$$R = [r(1)r(2)\cdots r(N-2)]^T$$

The matrix A is exactly the same as A of Eq. (2.2.4).

Table 1.3: The derivatives and integrals of $\varphi(x)$ ($N=6$)

x	$\varphi^{(1)}(x)$	$r(x)$
0.0	0.00000000E+00	0.00000000E + 00
0.5	0.15416762E+01	0.1413l460E+00
1.0	0.16384523E+01	0.60074157E+00
1.5	-0.24468283E+01	0. 10529082E+01
2.0	-0.22327582E+01	0.10967114E+01
2.5	0.12730265E+01	0.98506614E+00
3.0	0.55015936E+00	0.98548673E+00
3.5	-0.37227297E+00	0.10033183E+01
4.0	0.44146491E-01	0.99965909E+00
4.5	0.43985356E-02	0.99999151E+00
5.0	0.00000000E+00	0.10000000E+01

CHAPTER-3

3. Multiresolution in $L^2(R)$:

Multiresolution analysis is a formal approach to constructing orthogonal bases of wavelets, using a definite set of rules and procedure [2]. Multiresolution analysis consists of a sequence $\{V_j : j \in Z\}$ of embedded closed subspace of the space of square-integrable functions $L^2(R)$ that satisfies the following conditions:

I. $V_j \subset V_{j+1} \ \forall j \in Z$

II. $\bigcup_{j=-\infty}^{\infty} V_j R$ is dense in $L^2(R)$, i.e. , $\bigcup_{j=-\infty}^{\infty} V_j = L^2(R)$

III. $\bigcap_{j=-\infty}^{\infty} V_j = 0$

IV. $f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1}$`

V. If $\exists$ a function $\varphi \in V_0$ such that $\{\varphi_{0,n} = \varphi(x-n), n \in Z\}$ is an orthonormal basis for

V_0, i.e. , $\|f\|^2 = \int_{-\infty}^{\infty} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} \left|(f, \varphi_{0,n})\right|^2 \quad \forall f \in V_0$

The function φ is called scaling function or father wavelet. If $\{V_j\}$ is a multiresolution of square-integrable functions $L^2(R)$ and if V_0 is the closed subspace generated by the integer translates of a single function φ generates the multiresolution analysis.

The existence of a scaling function $\varphi(x)$ is required to generate a basis in each V_j by

$$V_j = \overline{span}\{\varphi_{ji}\}_{i \in Z} \tag{3.1}$$

with $$\varphi_{ij} = 2^{j/2} \varphi(2^j x - i), i, j \in Z$$

Let the W_j denote a subspace which complements the subspace V_j in V_{j+1}

i.e., $$V_{j+1} = V_j \oplus W_j .$$

Each element of V_{j+1} can be uniquely written as the sum of an element in V_j, and the element which contains the details required to pass from an approximation at level j to an approximation at level $j+1$ is in W_j.

Based on the function $\varphi(x)$one can construct the so-called mother wavelet $\psi(x)$ by

$$W_j = \overline{span}\{\psi_{ji}\}_{i \in Z} \tag{3.2}$$

with $$\psi_{ij} = 2^{j/2}\psi(2^j x - i), i, j \in Z$$

The scaling function $\varphi(x)$ and its mother wavelet $\psi(x)$ having the following properties:

- The area under the scaling function is unity

$$\int_{-\infty}^{\infty} \varphi(x)dx = 1 \tag{3.3}$$

- The scaling function and its translates are orthogonal

$$\int_{-\infty}^{\infty} \varphi(x-j)\varphi(x-i)dx = \delta_{i,j} \quad i, j \in Z \tag{3.4}$$

- The wavelet function has $L/2$ vanishing moments.

$$\int_{-\infty}^{\infty} x^m \psi(x)dx = o \quad m = 0,1,\cdots, L/2 \tag{3.5}$$

- The scaling and wavelet functions are orthogonal

$$\int_{-\infty}^{\infty} \varphi(x)\psi(x-k)dx = o \quad k \in Z \tag{3.6}$$

This condition implies that $V_{j+1} = V_j \oplus W_j$ on each fixed and scale j, the scaling functions $\{\varphi_{jk}(x)\}_{k \in Z}$ from an orthonormal basis V_j and the wavelets $\{\psi_{jk}(x)\}_{k \in Z}$ form an orthonormal basis W_j.

The set of spaces of set V_j is called as Multiresolution analysis of square-integrable functions, $L^2(R)$. These spaces will be used to approximate the solutions of Partial Differential Equations using the Wavelet-Galerkin method.

CHAPTER-4

4. Connection Coefficients:

In order to compute the solution of differential equation by using wavelet Galerkin method we need to compute the connection coefficients as explained in Latto et al. [6] as

$$\Omega_{l_1 l_2}^{d_1 d_2} = \int_{-\infty}^{\infty} \varphi^{d_1}(x - l_1)\varphi^{d_2}(x - l_2)dx \tag{4.1}$$

Taking derivatives of the scaling function d times, we get

$$\varphi^d(x) = 2^d \sum_{k=0}^{N-1} a_k \varphi^d(2x - k) \tag{4.2}$$

After simplifying and considering it for all $\Omega_{l_1 l_2}^{d_1 d_2}$, gives a system of linear

equations with $\Omega^{d_1 d_2}$ as unknown vector:

$$T\Omega^{d_1 d_2} = \frac{1}{2^{d-1}}\Omega^{d_1 d_2} \tag{4.3}$$

where $d = d_1 + d_2$ and $T = \sum_i a_i a_{q-2l+1}$.

The moments M_i^j of the translates of $\varphi(x)$ can be obtained by using the formula

$$M_i^j = \int_{-\infty}^{\infty} x^j \varphi(x - i)dx \tag{4.4}$$

with $M_0^0 = 1$

Latto et al. [1] derives the formula to compute the moments by induction on k .

$$M_i^j = \frac{1}{2(2^j - 1)} \sum_{k=0}^{j} \binom{j}{k} i^{j-k} \sum_{l=0}^{k-1} \binom{k}{l} M_l^0 \left(\sum_{i=0}^{N-1} a_i i^{k-l} \right) \tag{4.5}$$

where the $a_i's$ are the Daubechies wavelet coefficients.

Finally the system will be

$$\begin{bmatrix} T - \frac{1}{2^{d-1}} I \\ M^d \end{bmatrix} \Omega^{d_1 d_2} = \begin{bmatrix} 0 \\ d! \end{bmatrix}$$

Mathematical software is used to compute the moments and connection coefficients at different scales. The connection coefficients are computed by substituting the values of Daubechies coefficients in the matrix *T* and by evaluating moments by using the programme that is given in [12].

The connection coefficient at $j = 0, N = 6$ and $N = 12$ are given below.

Table 4.1: 2-term connection coefficients

Connection Coefficients at $N = 6$, $j = 0$, $d_1 = 2, d_2 = 0$

$\Omega[-4]$	5.357142857144194e-003
$\Omega[-3]$	1.142857142857108e-001
$\Omega[-2]$	-8.761904761904359e-001
$\Omega[-1]$	3.390476190476079e+000
$\Omega[0]$	-5.267857142857051e+000
$\Omega[1]$	3.390476190476190e+000
$\Omega[2]$	-8.761904761904867e-001
$\Omega[3]$	1.142857142857139e-001
$\Omega[4]$	5.357142857141956e-003

Connection Coefficients at $N = 12$, $j = 0$, $d_1 = 2, d_2 = 0$

$\Omega[-10]$	-1.26410E-11	$\Omega[10]$	-1.26410E-11
$\Omega[-9]$	2.62998E-08	$\Omega[9]$	2.62998E-08
$\Omega[-8]$	-3.46609E-08	$\Omega[8]$	-3.46609E-08
$\Omega[-7]$	-5.43634E-05	$\Omega[7]$	-5.43634E-05
$\Omega[-6]$	-6.56963E-05	$\Omega[6]$	-6.56963E-05
$\Omega[-5]$	6.47806E-03	$\Omega[5]$	6.47806E-03
$\Omega[-4]$	-4.93616E-02	$\Omega[4]$	-4.93616E-02
$\Omega[-3]$	2.04905E-01	$\Omega[3]$	2.04905E-01
$\Omega[-2]$	-6.30733E-01	$\Omega[2]$	-6.30733E-01
$\Omega[-1]$	2.31187E+00	$\Omega[1]$	2.31187E+00
$\Omega[0]$	-3.68606E+00		

CHAPTER-5

5. WAVELET METHODS FOR PDES:

5.1. Wavelet-Galerkin Solution of the Periodic Problem [13, 14]:

Consider the two dimensional Poisson's equation

$$u_{xx} + u_{yy} = f \tag{5.1.1}$$

where $u = u(x,y)$, $f = f(x,y)$ are periodic in x, y of period d_x, $d_y \in Z$

Let the approximate solution $u = u(x,y)$ at scale m be

$$u(x,y) = \sum_{k_1}\sum_{k_2} c_{k_1,k_2} 2^{m/2} \varphi(2^m x - k_1) 2^{m/2} \varphi(2^m y - k_2) \quad k_1, k_2 \in Z \tag{5.1.2}$$

where c_{k_1,k_2}'s are periodic wavelet coefficients of u.

Putting $X = 2^m x$, $Y = 2^m y$ so that Eq. (5.1.2) becomes

$$U(X,Y) = u(x,y) = \sum_{k_1}\sum_{k_2} c_{k_1,k_2} \varphi(X - k_1)\varphi(Y - k_2), C_{k_1,k_2} = c_{k_1,k_2} 2^{m/2} \tag{5.1.3}$$

$U(X,Y)$ and also C_{k_1,k_2} are periodic in X, Y with periods $n_x = 2^m d_x$, $n_y = 2^m d_y$. Let us discretize $U(X,Y)$ at all the dyadic points.

$$x = 2^{-m} X, \; y = 2^{-m} Y, \; X, Y \in Z$$

$$U_{i,j} = \sum_{k_1}\sum_{k_2} C_{k_1,k_2} \varphi_{i-k_1} \varphi_{j-k_2} = \sum_{k_1}\sum_{k_2} C_{k_1,k_2} \varphi_{k_1} \varphi_{k_2} \tag{5.1.4}$$

where $\quad i = 0,1,2,\ldots,n_x - 1$ and $j = 0,1,2,\ldots,n_y - 1$

The matrix representation is

$$U = k_{\varphi_x} * k_{\varphi_y} * C \tag{5.1.5}$$

where k_{φ_x}, k_{φ_y} are the convolution kernels, i.e. the first column of the scaling function matrices and C is the wavelet coefficient matrix.

Similarly, RHS of Eq. (5.1.1) can be expressed for

$$f(x,y) = \sum_{k_1}\sum_{k_2} d_{k_1,k_2} 2^{m/2} \varphi(2^m x - k_1) 2^{m/2} \varphi(2^m y - k_2) \quad k_1, k_2 \in Z \tag{5.1.6}$$

as

$$F(X,Y) = f(x,y) = \sum_{k_1}\sum_{k_2} d_{k_1,k_2} 2^{m/2} \varphi(X - k_1) 2^{m/2} \varphi(Y - k_2), D_{k_1,k_2} = d_{k_1,k_2} 2^{m/2} \tag{5.1.7}$$

takes as

$$F = k_{\varphi_x} * k_{\varphi_y} * D \tag{5.1.8}$$

Substituting $u(x,y)$ and $f(x,y)$ into the given Eq. (5.1.1) and taking inner product on both sides with $\varphi(X-n_1)$, $\varphi(Y-n_2)$, $n_1,n_2 \in Z$

Use $\Omega_{j,k} = \int \varphi^n(y-k)\varphi(y-j)dy$ and $\delta_{j,k} = \int \varphi(y-k)\varphi(y-j)dy$

We obtain, $$k_\Omega x C = \frac{1}{2^{2m}} g \tag{5.1.9}$$

Taking Fourier Transformation of Eqs. (5.1.5), (5.1.7) and (5.1.8), we get

$$\hat{U} = \frac{1}{2^{2m}} \frac{\hat{F}}{\hat{k}_\Omega} \tag{5.1.10}$$

Then inverse FT gives the solution U.

5.2. Capacitance Matrix Method and Boundary Conditions [13, 14]:

Consider the problem (5.1.1) defined over the region S with Dirichlet boundary conditions $u = u_\tau(x,y)$. Let u, f are periodic with period d_x, d_y. If f is not periodic, it can made periodic making it zero or extending smoothly outside the region S. Let $v(x,y)$ be the solution in S with periodic boundary conditions. Let the solution $u(x,y)$ with Dirichlet boundary conditions is obtained by adding another function $w(x,y)$ such that

$$u = v + w \tag{5.2.1}$$

Since $v_{xx} + v_{yy} = f$, w must satisfy $w_{xx} + w_{yy} = 0$ in the region S.

However, on or outside τ, $\nabla^2 w = w_{xx} + w_{yy}$ may take such values as to make u satisfy the given Dirichlet boundary conditions. The desired effect is achieved by placing delta function along a closed boundary τ_1 which encompasses the region S. In other words, the solution w is given by

$$w_{xx} + w_{yy} = X_1 \text{ in } [0,d_x],[0,d_y] \tag{5.2.2}$$

Where $$X_1 = X(x,y) = \int_{\tau_1} X_0(p,q)\delta(x-p,y-q)d\tau_1$$

And $\delta(x,y)$ stands for delta function at $(0,0)$. So the solution

$$w(x,y) = G(x,y) * X_1(x,y) = \int_{\tau_1} X_0(p,q)G(x-p,y-q)d\tau_1, \quad (p,q) \in \tau_1 \tag{5.2.3}$$

Discretize (5.2.3) at the points (p_j,q_j), $j = 1,2,\ldots,n_\tau$ on π_1. n_τ stands for number of points on τ.

$$w(x,y) = \sum_j X_0(p_j,q_j)G(x-p_j,y-q_j) \tag{5.2.4}$$

Similarly considering the mesh points on $(x_i, y_j),\ i = 1,2,\ldots,n_\tau$ on τ . The discretized solution of Eq. (5.2.4) takes the form

$$w_i \equiv w(x_i, y_i) = \sum_j X_0(p_j, q_j) G(x_i - p_j, y_i - q_j) \tag{5.2.5}$$

Solving the matrix $w = GX$

where $\quad w = [w_i],\ G = [G_{i,j}] = G(x_i - p_j, y_i - q_j),\ X = [X_i]$

We get X_i which substituted in (5.2.4), gives w.

5.3. Wavelet –Galerkin Fictitious Boundary Approach [13]:

Consider the two dimensional Poisson's equation

$$u_{xx} + u_{yy} = f \tag{5.3.1}$$

Let the approximate solution be

$$u(x, y) = \sum_{k_1} \sum_{k_2} c_{k_1,k_2} 2^{m/2} \varphi(2^m x - k_1) 2^{m/2} \varphi(2^m y - k_2) \quad k_1, k_2 \in Z$$

$$= \sum_{k_1} \sum_{k_2} C_{k_1,k_2} \varphi(X - k_1) \varphi(Y - k_2) \tag{5.3.2}$$

where $\quad C_{k_1,k_2} = c_{k_1,k_2} 2^{m/2},\ X = 2^m x,\ Y = 2^m y$

Similarly

$$f(x, y) = \sum_{k_1} \sum_{k_2} D_{k_1,k_2} \varphi(X - k_1) \varphi(Y - k_2) \tag{5.3.3}$$

where $\quad D_{k_1,k_2} = d_{k_1,k_2} 2^{m/2}$

Substitute $u(x, y)$ and $f(x, y)$ into the given Eq. (5.3.1) and then take inner product on both sides with $\varphi(X - n_1), \varphi(Y - n_2),\ n_1, n_2 \in Z$

$$\sum_{k_1} \sum_{k_2} 2^{2m} C_{k_1,k_2} \int \varphi''(X - k_1) \varphi(X - n_1) dx \int \varphi(Y - k_2) \varphi(Y - n_2) dy +$$

$$\sum_{k_1} \sum_{k_2} 2^{2m} C_{k_1,k_2} \int \varphi(X - k_1) \varphi(X - n_1) dx \int \varphi''(Y - k_2) \varphi(Y - n_2) dy \tag{5.3.4}$$

$$= \sum_{k_1} \sum_{k_2} D_{k_1,k_2} \int \varphi(X - k_1) \varphi(X - n_1) dx \int \varphi(Y - k_2) \varphi(Y - n_2) dy$$

or

$$\sum_{k_1} C_{k_1,n_2} \int \varphi''(X - k_1) \varphi(X - n_1) dx + \sum_{k_2} C_{n_1,k_2} \int \varphi''(Y - k_2) \varphi(Y - n_2) dy = \frac{1}{2^{2m}} D_{n_1,k_2} \tag{5.3.5}$$

i.e. , $$\sum_{k_1} C_{k_1,n_2}\Omega_{n_1-k_1} + \sum_{k_2} C_{n_1,k_2}\Omega_{n_2-k_2} = \frac{1}{2^{2m}} D_{n_1,k_2} \tag{5.3.6}$$

where $\Omega_{n_1-k_1} = \int (X-k_1)\varphi(X-n_1)dx$ and $\Omega_{n_2-k_2} = \int \varphi(Y-k_2)\varphi(Y-n_2)dy$

are the 2-term connection coefficient.

Solving Eq. (5.3.6) will give the wavelet coefficients and hence the solution.

CHAPTER-6

6. TEST PROBLEM:

Problem: Consider 1D Wave Equation

$$u_{xx} + \alpha\, u = 0 \qquad\qquad \alpha > 0, 0 \le x \le 1 \tag{6.1}$$

with Dirichlet boundary condition $u(0) = 1$and $u(1) = 0$, $\alpha = 2\times 10^{-6}$

Exact solution of Eq. (6.1) is $u = \cos\sqrt{\alpha}\, x - \cot\sqrt{\alpha}\, \sin\sqrt{\alpha}\, x$

Here we consider $N = 6$ and $j = 0$

Let the approximated solution of Eq. (6.1) be

$$u(x) = \sum_{k=1-N}^{2^j} c_k\, 2^{j/2}\, \varphi(2^j - k)$$

$$= \sum_{k=1-N}^{2^j} C_k \varphi(X - k) \tag{6.2}$$

where $\qquad C_k = c_k\, 2^{j/2}$ and $X = 2^j x$

Substituting $u(x)$ in the given Eq. (6.1) and then take inner product with $\varphi(X - n\ ), n \in Z$, we get

$$\sum_k C_k \Omega_{k,n} + \alpha \sum_k C_k \delta_{k,n} = 0 \tag{6.3}$$

$$n = 1 - L, 2 - L, \ldots, 2^j$$

i.e.; $\qquad n = -5, -4, \ldots, 0, 1$

where $\qquad \Omega_{k,n} = \int \varphi''(X - k)\varphi(X - n)$

and $\qquad \delta_{k,n} = \int \varphi(X - k)\varphi(X - n)$

By using Dirichlet boundary conditions in Eq. (6.2), we get

$$u(0) = \sum_{k=-5}^{1} C_k \varphi(-k) = 1 \tag{6.4}$$

and $$u(1) = \sum_{k=-5}^{1} C_k \varphi(1 - k) = 0 \tag{6.5}$$

From the Dirichlet left boundary condition, we get Eq. (6.4) and from right boundary condition, we get Eq. (6.5), which replace the first and last equation of Eq. (6.3) respectively. After replacing the two rows from Eq. (6.3), we get the following matrix with $N = 6$.

$$TC = B$$

$$T = \begin{bmatrix} 0 & \varphi(4) & \varphi(3) & \varphi(2) & \varphi(1) & 0 & 0 \\ \Omega_1 & \Omega_0 + \alpha & \Omega_{-1} & \Omega_{-2} & \Omega_{-3} & \Omega_{-4} & \Omega_{-5} \\ \Omega_2 & \Omega_1 & \Omega_0 + \alpha & \Omega_{-1} & \Omega_{-2} & \Omega_{-3} & \Omega_{-4} \\ \Omega_3 & \Omega_2 & \Omega_1 & \Omega_0 + \alpha & \Omega_{-1} & \Omega_{-2} & \Omega_{-3} \\ \Omega_4 & \Omega_3 & \Omega_2 & \Omega_1 & \Omega_0 + \alpha & \Omega_{-1} & \Omega_{-2} \\ \Omega_5 & \Omega_4 & \Omega_3 & \Omega_2 & \Omega_1 & \Omega_0 + \alpha & \Omega_{-1} \\ 0 & 0 & \varphi(4) & \varphi(3) & \varphi(2) & \varphi(1) & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} C_{-5} \\ C_{-4} \\ C_{-3} \\ C_{-2} \\ C_{-1} \\ C_0 \\ C_1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now applying Gauss elimination method and using mathematical software we get the values of C_k. The values of C_k's are

C_{-5} = 3.8385

C_{-4} = 4.3212

C_{-3} = 3.4306

C_{-2} = 2.3060

C_{-1} = 1.2008

C_0 = 0.1781

C_1 = -0.4506

The solution is obtained by directly substituting the value of C_k in the Eq. (6.2).

Table 6.1: Comparison between Wavelet Solution and Exact solution

x	Wavelet solution	Exact solution	Absolute error
0	1.0000138607	1	0.00001
.125	0.8727584879	0.8750000684	0.00224
.25	0.7470655427	0.7500001094	0.00293
.375	0.6195812986	0.625000127	0.00541
.5	0.4956624395	0.500000125	0.00433
.625	0.3693354948	0.3750001074	0.00566
.75	0.2446498173	0.2500000781	0.00053
.875	0.1220800624	0.125000041	0.00291
1	0	0	0

From the above we conclude that Wavelet-Galerkin method yields better results which shows the efficiency of this method.

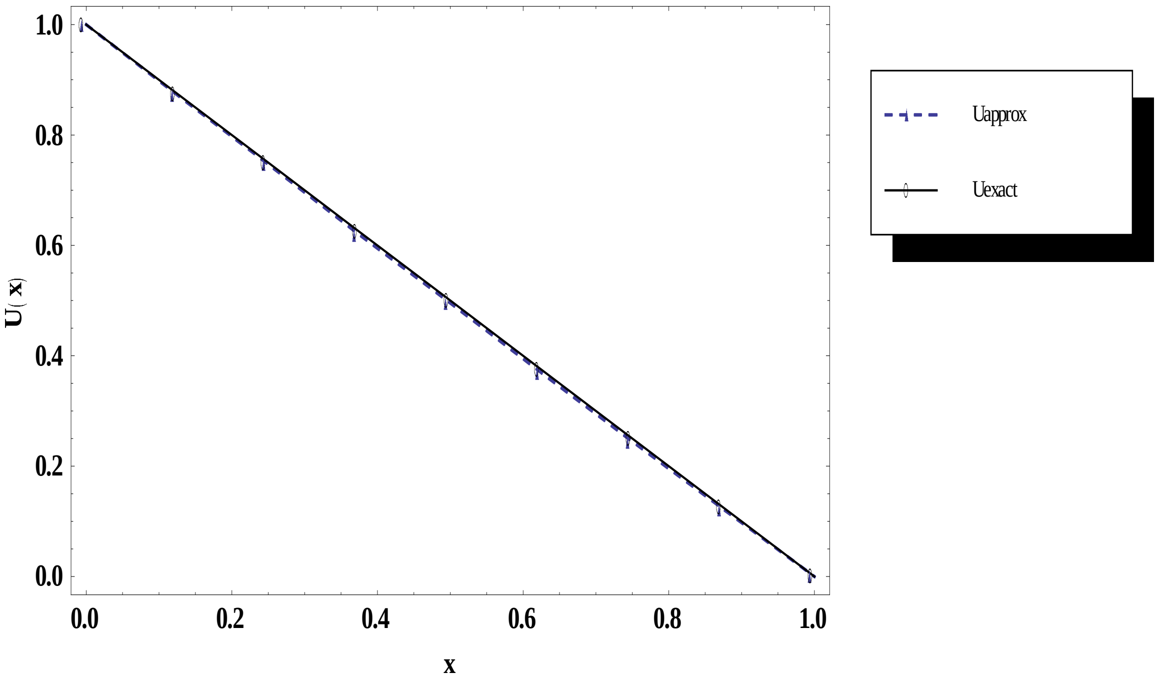

Figure 1: Comparison between Wavelet solution and Exact solution.

In this above graph Uapprox represents the Wavelet-Galerkin solution and Uexact represents the exact solution.

CHAPTER-7

CONCLUSION

The Wavelet-Galerkin method has been shown to be a powerful and efficient numerical tool for fast and more accurate solution of differential equations. For the difficulty of the boundary conditions we offered the idea of replacing the first and last equations by the equations obtained from the left and right boundaries in the one dimensional problems. The connection coefficient can be computed analytically which leads to stability of the wavelet –based method. The Daubechies' order N provides more smoothness of the scaling function and a large dilation order j gives finer resolution.

An obtained advantages of the Galerkin method is it uses Daubechies' coefficients and compute the scaling function, the connection coefficients and the rest of the components only once. This leads to save a considerable computational time and improves the numerical results through the reduction of rounding off errors.

Wavelet-Galerkin method is the most frequently used scheme due to its fast and accurate solutions of differentially equations, which can be observed from the result. Wavelet Galerkin method yields better results which shows the efficiency of this method.

Solution obtained using Daubechies-6 coefficients wavelet has been compared with the exact solution. The results obtained comparing with the exact solution proves the accuracy and efficiency of this method.

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